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THE ASYMPTOTIC FORM OF THE SPECTRUM OF TIDAL KELVIN WAVES TRAPPED BY A BOUNDARY AT LARGE VALUES OF THE LAMB PARAMETER†

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Tidal Laplace equations are considered and the asymptotic form of the eigenvalues corresponding to eigenfunctions (Kelvin waves) concentrated in the neighbourhood of a closed component of the boundary are constructed for large values of the Lamb parameter, which is proportional to the square of the angular velocity of rotation of the reservoir. The formulae obtained determine the asymptotic behaviour of the kth eigenvalue in the case of a planar, simply connected reservoir of constant depth. © 1999 Elsevier Science Ltd. All rights reserved.

1. TIDAL LAPLACE EQUATIONS

Suppose X is a bounded, smooth, orientated surface with an edge and $C^\infty(X)$ and $\Lambda_1(X)$ are sets of smooth complex-valued functions and covector fields in X. We put

$$\langle U,V\rangle = g^{ij}U_i\bar{V}_j,\ (*U)_i = \varepsilon_{ij}(\det g)^{1/2}g^{ij}U_k$$

$$\operatorname{div} U = (\det g)^{-1/2}\frac{\partial}{\partial x_i}((\det g)^{1/2}g^{ij}U_j),\ U,V\in\Lambda_1(X)$$

Here g^{ij} are the covariant components of the metric surface tensor, $\det g$ is the determinant of the matrix $\{g_{ij}\}$ (which is the inverse of $\{g^{ij}\}$) and $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. Summation over repeated indices is implied.

Suppose the space $\mathbb{R}^3$ is rigidly connected with the rotating surface X and that $(0, 0, \Omega)$ is the vector of the angular velocity of rotation and n is the field of the normals to the surface X matched with its orientation. The latter means that the vectors (r_{x^1}, r_{x^2}, n), where $r = r(x^1, x^2)$ is the parametrization of the surface, form a right-hand triad.

We now consider a particle of unit mass which moves over the surface X with an instantaneous velocity which is determined by the covector U. Direct verification in local coordinates shows that the expression for the Coriolis force acting on it can be rewritten as $-2\Omega pn\times U = 2\Omega p * U$. Here $p = \cos\theta$, where θ is the angle between the normal n and the axis of rotation, $\times$ is vector multiplication, and the natural identification of the covectors and the tangential vectors to the surface lying in $\mathbb{R}^3$: $U\mapsto g^{ij}U_j r_{x^i}$ is implied.

In view of the above relation the tidal Laplace equations [1, 2] take the form

$$i\lambda U = p*U - \alpha^{-1/2}h\nabla\zeta,\ i\lambda\zeta = -\alpha^{-1/2}\operatorname{div} U \tag{1.1}$$

Here $\lambda = \omega/(2\Omega)$ is a spectral parameter, $\alpha = 4\Omega^2 l_*^2(gh)$ is a dimensionless parameter (the Lamb parameter), ω is the frequency of the free oscillations, l is a characteristic dimension, h is the characteristic depth and g is the acceleration due to gravity. The dimensionless unperturbed depth of the fluid $h\in C^\infty(X)$ is assumed to be positive: $\inf_X h > 0$. The quantities (U, ζ) determine the velocity flux and the elevation of the free surface.

We define the Hilbert spaces L_2', $H_{1,h}'$ as the completion of the sets $\{\varphi\in C^\infty(X) : \int\varphi dS = 0\}$ using the norms which are generated by the scalar products

$$(\varphi,\psi)_0 = \int\varphi\bar{\psi}dS,\ (\varphi,\psi)_{1,h} = \int h\langle\nabla\varphi,\nabla\psi\rangle dS$$

and the space $\hat{L}_{2,h}$ as the completion of $\Lambda_1(X)$ using the norm associated with the scalar product $[U, V]_{0,h} = \int h^{-1}\langle U, V\rangle dS$. Here, dS is an element of area and integration is carried out over the surface X.

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The set of elements $U \in \hat{L}_{2,h}$ for which $\operatorname{div} U \in L_2(X, dS)$ is denoted by $\hat{E}$. The sign γ of the normal component is ∂X is correctly defined for the elements of $\hat{E}$ (compare with [3, p. 101]). If $U \in \Lambda_1$, then $-\gamma U$ conforms with the contraction $*U$ in ∂X. We put $\hat{E}_0 = \{U \in \hat{E}: \gamma_v U = 0\}$.

The eigenvalue problem for the tidal Laplace equations (1.1) with an impermeability condition on the boundary reduces to an investigation of the eigenvalue properties of the operator

$$\tilde{R} = \begin{Vmatrix} -ip* & i\alpha^{-1/2} h\nabla \\ i\alpha^{-1/2}\operatorname{div} & 0 \end{Vmatrix} : \hat{L}_{2,h} \oplus L_2' \to \hat{L}_{2,h} \oplus L_2'$$

in the domain of definition $\tilde{R} = \hat{E}_0 \otimes H'_{1,h}$.

The operator $\tilde{R}$ is self-adjoint. The structure and asymptotic form of its spectrum are described in Theorem 1 from [4]. Here, we note that the spectrum is symmetrical with respect to the origin of the coordinate system and consists of the point $\lambda = 0$ and a denumerable set of eigenvalues of finite multiplicity with the limit points $\pm\infty$ when $ph^{-1} \equiv$ const and 0, $\pm\infty$ when $ph^{-1} \not\equiv$ const. Variational principles for the eigenvalues have been formulated earlier [2, 4, 5].

2. ASYMPTOTIC BEHAVIOUR OF THE KELVIN WAVE SPECTRUM

The asymptotic form of the eigenvalues corresponding to the eigenfunctions concentrated in the neighbourhood of the boundary are constructed using the boundary-layer method [6, 7]. Suppose that Γ_0 is a component of the boundary of the surface X of length l. In a certain neighbourhood of this component, we introduce a semigeodesic system of coordinates $(\tau, s) \in [0, l] \times [0, \varepsilon_0)$, where τ is the length of the arc Γ_0 and s is the length of the arc of the geodesic which is orthogonal to Γ_0. As τ increases, the surface remains to the left.

In the coordinates (τ, s), we have $g_{11} = G(\tau, s), g_{12} = g_{21} = 0, g_{22} = 1, G(\tau, 0) = 1$. Putting $U = U_1 d\tau + U_2 ds$, we rewrite Eqs (1.1) in the form

$$i\lambda U_1 = pG^{1/2}U_2 - \alpha^{-1/2}h\frac{\partial\zeta}{\partial\tau}, \quad i\lambda U_2 = -pG^{-1/2}U_1 - \alpha^{-1/2}h\frac{\partial\zeta}{\partial s} \tag{2.1}$$

$$i\lambda\zeta = -\alpha^{-1/2}G^{-1/2}\left(\frac{\partial}{\partial\tau}(G^{-1/2}U_1) + \frac{\partial}{\partial s}(G^{1/2}U_2)\right)$$

We now define the stretched variable $\rho = \alpha^{1/2}s$ and seek the asymptotic form of the eigenvalues and the eigenfunctions in the form

$$\lambda \sim \alpha^{-1/2}\sum_{m=0}^{\infty}\delta_m\alpha^{-m/2}, \quad \zeta \sim \sum_{m=0}^{\infty}\zeta_m(\tau,\rho)\alpha^{-m/2} \tag{2.2}$$

$$U_1 \sim \sum_{m=0}^{\infty}U_{1,m}(\tau,\rho)\alpha^{-m/2}, \quad U_2 \sim \alpha^{-1/2}\sum_{m=0}^{\infty}U_{2,m}(\tau,\rho)\alpha^{-m/2}$$

where the functions ζ_m, $U_{j,m}$, which can be taken as being defined in the half-plane $(\tau, \rho) \in [0, l] \times [0, +\infty)$, decay exponentially when $\rho \to +\infty$ and are l-periodic with respect to τ. Furthermore, it is assumed that $\delta_0 > 0$.

We substitute expressions (2.2) into Eqs (2.1) and equate the coefficients of like powers of α. In the principal approximation we have

$$i\delta_0 U_{1,0} - p_0 U_{2,0} + h_0\frac{\partial\zeta_0}{\partial\tau} = 0, \quad p_0 U_{1,0} + h_0\frac{\partial\zeta_0}{\partial\rho} = 0 \tag{2.3}$$

$$i\delta_0\zeta_0 + \frac{\partial U_{1,0}}{\partial\tau} + \frac{\partial U_{2,0}}{\partial\rho} = 0 \tag{2.4}$$

Here, $h_0 = h(\tau, 0), p_0 = p(\tau, 0)$. It follows from the impermeability condition $*U|_{\Gamma_0} = 0$ that

$$U_{2,m}(\tau,0) = 0, \quad m \geqslant 0 \tag{2.5}$$

Suppose that $p_0 \neq 0$ in Γ_0. If X is a domain on a sphere which is symmetrical about the axis of rotation, then this means that Γ_0 does not intersect with the equator. On expressing the functions $U_{j,0}$ from Eqs (2.3) in terms of ζ_0 and substituting into (2.4) and (2.5), we obtain

$$\frac{\partial^2\zeta_0}{\partial\rho^2}-i\frac{p_0^2}{\delta_0 h_0}\frac{d}{d\tau}\left(\frac{h_0}{p_0}\right)\frac{\partial\zeta_0}{\partial\rho}-\frac{p_0^2}{h_0}\zeta_0=0 \tag{2.6}$$

$$\frac{\partial\zeta_0}{\partial\tau}(\tau,0)-i\frac{\delta_0}{p_0}\frac{\partial\zeta_0}{\partial\rho}(\tau,0)=0 \tag{2.7}$$

For sufficiently large δ_0, the characteristic polynomial of Eq. (2.6) has a root $\mu=\mu(\tau)$ with a negative real part. The corresponding solution of Eq. (2.6), which decays exponentially as $\rho\to+\infty$, has the form $\zeta_0=a_0(\tau)\exp(\mu\rho)$. On substituting this expression into (2.7), we obtain an ordinary first-order differential equation for $a_0=a_0(\tau)$. The condition of the l-periodicity of the function a_0 leads to the relations

$$\operatorname{sign}(p_0)\int_0^l\left[\frac{\delta_0^2}{h_0}-\frac{p_0^2}{4h_0^2}\left(\frac{d}{d\tau}\frac{h_0}{p_0}\right)^2\right]^{1/2}d\tau=2\pi k \tag{2.8}$$

where k is an integer. Equation (2.8) has a unique solution $\delta_0=\delta_0(k)$ for any sufficiently large k ($\operatorname{sign}k=\operatorname{sign}p_0$).

The corrections $(U_{1,m}, U_{2,m}, \zeta_m)$, δ_m, $m\geqslant 1$ are determined using the same scheme.

Using the well-known estimate of the distance to the spectrum Σ of a self-adjoint operator

$$\operatorname{dist}(\lambda;\Sigma(\tilde R))\leqslant\left\|(\lambda I-\tilde R)(U,\zeta);L_2'\oplus\hat L_{2,h}\right\|/\left\|(U,\zeta);L_2'\oplus\hat L_{2,h}\right\|$$

and following the well-known approach [9, p. 188], we deduce that the distance from the quantity $\lambda_N=\alpha^{-1/2}(\delta_0+\delta_1\alpha^{-1/2}+\ldots+\delta_N\alpha^{-N/2})$ to $\Sigma(\tilde R)$ is of the order of $\alpha^{-(N+1)/2}$. By virtue of the discreteness of the spectrum, this means that a denumerable set of eigenvalues $\lambda^k=\lambda^k(\alpha)$ exists, the asymptotic form of which is determined by the first formula of (2.2) and the coefficient $\delta_0(k)$ is calculated using formula (2.8).

Elementary analysis of the approximate solution which has been constructed shows that waves $\exp(i\lambda^k t)\zeta_0$, where t is the time, propagate along the boundary while the domain remains to the left if $\operatorname{sign}(p_0)>0$ (the northern hemisphere in the case of a sphere) and to the right if $\operatorname{sign}(p_0)<0$ (the southern hemisphere). When $h_0\equiv p_0\equiv 1$ the principal term of the asymptotic form for their velocity of propagation $d\tau/dt\sim\alpha^{-1/2}$ in dimensioned variables is the same as the velocity of the long waves $\surd(g_*h_*)$. Both of these properties are characteristic of Kelvin waves [10, 11].

We note that the compactness of the surface X does not play any role in the formal algorithm which has been described. Without considering issues concerning the substantiation of the asymptotic form of the spectrum in the case of a non-compact surface, we note that formula (2.8) for the exterior of a circle (when $h\equiv p\equiv 1$) has been obtained earlier [12].

3. THE ASYMPTOTIC FORM OF THE kth EIGENVALUE IN THE CASE OF A PLANAR, SIMPLY CONNECTED RESERVOIR OF CONSTANT DEPTH

Suppose X is a planar, simply connected, boundary domain, l is the length of its boundary and $h\equiv p\equiv 1$. It turns out that, in this case, the formulae which have been obtained determine the asymptotic form of the kth eigenvalue in the ordered sequence

$$0<\lambda_1\leqslant\lambda_2\leqslant\ldots\lambda_k\leqslant\ldots \tag{3.1}$$

of eigenvalues of the operator $\tilde R$, that is

$$\lambda_k\sim\frac{2\pi k}{l}\alpha^{-1/2}+\sum_{m=1}^{\infty}\delta_m(k)\alpha^{-(m+1)/2},\quad\alpha\to\infty \tag{3.2}$$

The scheme for the proof of this assertion is as follows. Suppose the boundaries of the unit circle $X_0=\{(x,y):x^2+y^2<1\}$ and of the domain X are connected by a smooth homotopy [13] and that

$\varepsilon \in [0, T]$ is the parameter of the homotopy and the boundaries of the domains $X(\varepsilon)$ obtained in the homotopy process have a length $l(\varepsilon)$. We denote the eigenvalues (3.1), corresponding to a domain $X(\varepsilon)$, by $\lambda_k(\alpha; \varepsilon)$ and we denote by $\psi = \psi(z; \varepsilon, \varepsilon_1)$, $z = x + iy$, the family of conformal mappings of the domain $X(\varepsilon_1)$ onto $X(\varepsilon)$, selected in such a way that

$$f_1(\varepsilon,\varepsilon_1) = \min_{z\in X(\varepsilon_1)} |\psi'_z(z;\varepsilon,\varepsilon_1)|^2,\quad f_2(\varepsilon,\varepsilon_1) = \max_{z\in X(\varepsilon_1)} |\psi'_z(z;\varepsilon,\varepsilon_1)|^2$$

are continuous functions of ε and $f_j(\varepsilon_1, \varepsilon_1) = 1, j = 1, 2$.

On analysing the known solution [1], we find the principal term of the asymptotic form (3.2) in the case of a unit circle. It follows from the result obtained and the inequality

$$\lambda_k(\alpha f_2(\varepsilon,\varepsilon_1);\varepsilon_1) \leqslant \lambda_k(\alpha;\varepsilon) \leqslant \lambda_k(\alpha f_1(\varepsilon,\varepsilon_1);\varepsilon_1) \tag{3.3}$$

which is derived using variational methods, that the bounded functions

$$F_{1,k}(\varepsilon) = \underline{\lim}_{\alpha\to\infty}\alpha^{1/2}\lambda_k(\alpha;\varepsilon),\quad F_{2,k}(\varepsilon) \equiv \overline{\lim}_{\alpha\to\infty}\alpha^{1/2}\lambda_k(\alpha;\varepsilon)$$

are defined.

Starting from the fact, established in the previous section, that an eigenvalue with the asymptotic form (3.2) exists and inequalities (3.3), it can be shown that the set

$$D_k = \{\varepsilon\in[0,T] : F_{1,k}(\varepsilon) = F_{2,k}(\varepsilon) = 2\pi k / l(\varepsilon)\}$$

is open and closed in $[0, T]$. Since D_k is not empty: $0 \in D_k$, it then follows from this that $D_k = [0, T]$.

Thus, the formula has been proved for the leading term of the asymptotic form of λ_k. The existence of an eigenvalue with the asymptotic form (3.2) shows that the complete expansion holds.

Note that the asymptotic form of a number of eigenvalues in a fixed interval $(0, \lambda)$, when $\alpha \to \infty$, have been found earlier in [5]. A detailed account of these and other results is available.†

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